The slender elliptic cone as a model for non-linear supersonic flow theory

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Summary

The second-order slender-body solution is derived for an unyawed elliptic cone in supersonic flow. The result is used as the basis for a critique of various approximations in compressible flow theory: slender-body, linearized, first- and second-order thin-wing theories; edge corrections; and the method of linearized characteristics.

1. Introduction

The circular cone serves as a standard of comparison for supersonic flow past bodies of revolution, and the elliptic cone will probably assume the same role for non-circular shapes. 'Exact' numerical solutions for elliptic cones comparable with those available for circular cones would require elaborate computing programmes that are not yet contemplated. However, a number of approximate theories for the elliptic cone have recently been advanced, based upon various linearizing assumptions. It can be anticipated that attempts will be made to improve some of these by successive approximations, so as to take account of the non-linear nature of compressible flow.

In this paper a second approximation to the supersonic flow past unyawed elliptic cones is obtained by proceeding from the slender-body theory. The first approximation, following Ward's (1949) definitive treatment of supersonic slender-body theory, has been given by Fraenkel (1952). The principles of deriving the second approximation by iteration have been set forth by Lighthill (1954) and Van Dyke (1956), the present treatment of elliptic cones being the first application to non-circular bodies. Whereas Adams & Sears' 'not-so-slender-body' theory (1953) seeks only a closer approach to the solution of the linearized Prandtl–Glauert equation, second-order slender-body theory includes also the leading non-linear terms.

Although the solution given here has intrinsic interest, it is utilized primarily as a model for the full inviscid solution, and thus serves as the basis for a critique of various approximations commonly employed in compressible flow theory. Thus the second-order solution is regarded as being exact (since in one respect or another it is indeed more exact than any of the approximations to be considered), and the approximation to be tested is introduced in addition to the simplifications already made.

2. The second-order slender-body solution

2.1. *Résumé of the first-order solution*

Consider a uniform supersonic stream of Mach number M flowing along the z-axis. Introduce the non-orthogonal elliptic-conical coordinates (ξ, η, s) by setting

$$\left.\begin{aligned} x+iy &= cs\cosh(\xi+i\eta), \\ z &= s. \end{aligned}\right\} \tag{1}$$

Then an unyawed elliptic cone is described by $\xi=\xi_0$ or (figure 1)

$$\left.\begin{aligned} x &= cs\cosh\xi_0\cos\eta = as\cos\eta, \\ y &= cs\sinh\xi_0\sin\eta \ = bs\sin\eta, \\ c^2 &= a^2-b^2. \end{aligned}\right\} \tag{2}$$

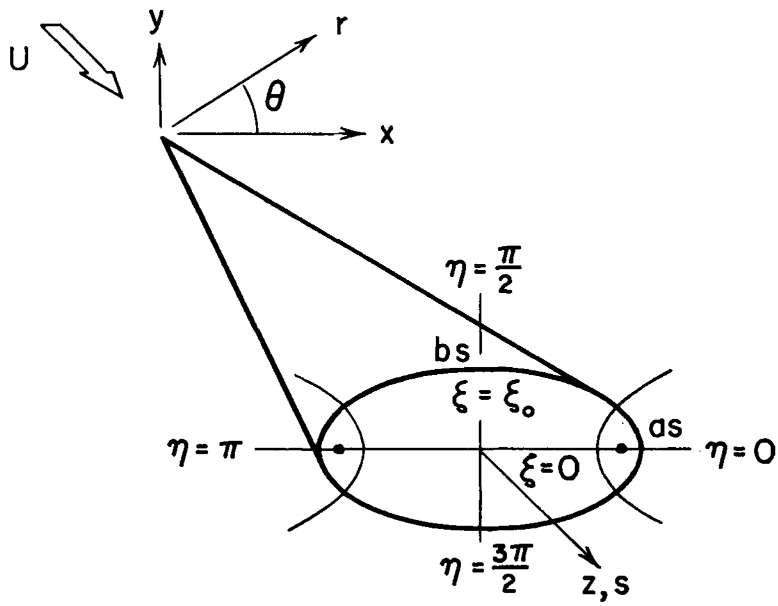

Figure 1. Notation for elliptic cone.

To second order the flow is irrotational, so that the velocity is the gradient of a potential Φ. Introduce a normalized perturbation potential ϕ by setting $\Phi=U(z+\phi)$ where U is the free-stream speed. Then the equation of motion, including all terms whose effect may be of second order in the slender-body approximation, is

$$\begin{aligned}\phi_{xx}+\phi_{yy} = \beta^2\phi_{zz} &+ 2M^2(\phi_x\phi_{xz}+\phi_y\phi_{yz})+(\gamma+1)M^4\phi_z\phi_{zz} \\ &+ M^2(\phi_x^2\phi_{xx}+2\phi_x\phi_y\phi_{xy}+\phi_y^2\phi_{yy}),\end{aligned} \tag{3}$$

where $\beta^2=M^2-1$. The boundary conditions are that the flow be tangential to the surface and that, to second order, the perturbation potential vanish at the Mach cone $r=(x^2+y^2)^{1/2}=\beta z$.

In R. T. Jones's slender-wing theory (1946) all terms on the right-hand side of (3) are neglected, although for slender bodies with thickness the term

$\beta^2\phi_{zz}$ is implicit in the condition far from the body, as shown by Ward (1949). Hence in elliptic-conical coordinates the slender-body equation is simply

$$\phi_{\xi\xi}+\phi_{\eta\eta}=0. \tag{4}$$

The condition of tangential flow at the surface is found to be

$$\phi_\xi = abs(1+\phi_z) \text{ at } \xi=\xi_0, \tag{5}$$

where ϕ_z can be neglected in linearized, and hence also slender-body, theory. The general solution of these equations, together with its asymptotic behaviour far from the body, is

$$\phi = abs\xi + C \sim abs\log\frac{2r}{cs}+C. \tag{6}$$

The constant C, which may depend upon s, is evaluated by considering the asymptotic behaviour. Ward's theory provides a general connection between the term proportional to $\log r$ and that independent of r, which for conical bodies is that the perturbation potential should be asymptotically proportional to

$$1+\log\frac{\beta r}{2s}.$$

Thus the slender-body solution is found to be

$$\phi = abs\left(\xi+1+\log\frac{\beta c}{4}\right). \tag{7}$$

First derivatives in cartesian and elliptic-conical coordinates are related on the surface of the cone by

$$\left.\begin{aligned}\phi_x &= \frac{b\cos\eta\,\phi_\xi - a\sin\eta\,\phi_\eta}{s(a^2\sin^2\eta+b^2\cos^2\eta)},\\ \phi_y &= \frac{a\sin\eta\,\phi_\xi + b\cos\eta\,\phi_\eta}{s(a^2\sin^2\eta+b^2\cos^2\eta)},\\ \phi_z &= \phi_s + \frac{(a^2-b^2)\sin\eta\cos\eta\,\phi_\eta - ab\phi_\xi}{s(a^2\sin^2\eta+b^2\cos^2\eta)}.\end{aligned}\right\} \tag{8}$$

The pressure coefficient is given to second order by

$$C_p = -2\phi_z - (\phi_x^2+\phi_y^2) + \beta^2\phi_z^2 + M^2\phi_z(\phi_x^2+\phi_y^2) + \tfrac{1}{4}M^2(\phi_x^2+\phi_y^2)^2, \tag{9}$$

of which only the first two terms are required in linearized theory. Hence the slender-body approximation to the pressure coefficient at the surface of the elliptic cone is found to be

$$C_p = ab\left[2\log\frac{4}{\beta(a+b)} - 2 + \frac{ab}{a^2\sin^2\eta+b^2\cos^2\eta}\right]. \tag{10}$$

The drag coefficient (referred to base area) is the average with respect to η of the pressure coefficient:

$$C_D = \frac{1}{2\pi}\int_0^{2\pi} C_p\,d\eta = ab\left[2\log\frac{4}{\beta(a+b)}-1\right]. \tag{11}$$

These results were given by Fraenkel (1952); equivalent results of greater complexity were obtained by Kahane & Solarski (1953).

2.2. *The not-so-slender solution*

The slender-body solution is now to be refined by iteration. Rather than attack at once all the terms neglected on the right-hand side of (3), consider first only the single linear term $\beta^2\phi_{zz}$. Thus a second approximation is sought to the solution of the Prandtl–Glauert equation, which may be written for purposes of iteration as

$$\phi_{xx}+\phi_{yy}=\beta^2\phi_{zz}. \tag{12}$$

Evaluation of the right-hand side in terms of the previous first approximation yields, in elliptic-conical coordinates,

$$\phi_{\xi\xi}+\phi_{\eta\eta}=\tfrac{1}{2}\beta^2 ab(a^2-b^2)s\sinh 2\xi\,\frac{3\sin^2 2\eta-\sinh^2 2\xi}{(\cosh 2\xi-\cos 2\eta)^2}\,. \tag{13}$$

The usual technique of introducing $(\xi+i\eta)$ and $(\xi-i\eta)$ as independent variables permits a particular integral ψ to be found directly by integration:

$$\psi=\tfrac{1}{8}\beta^2 ab(a^2-b^2)s\sinh 2\xi\left(\frac{2\cos 2\eta}{\cosh 2\xi-\cos 2\eta}-1\right). \tag{14}$$

It can be verified from the general theory (Lighthill 1954, Van Dyke 1956) that this particular integral behaves asymptotically in the manner appropriate to vanishing perturbation potential at the Mach cone. Hence the condition far from the body is satisfied provided that the complementary function required to restore tangential flow at the surface vanishes at infinity, except possibly for a multiple of the first approximation (7).

The linearized tangency condition has already been satisfied in the first approximation, so the correction consisting of the particular integral ψ plus the complementary function χ must satisfy $\psi_\xi+\chi_\xi=0$ at $\xi=\xi_0$. Possible complementary functions that vanish far from the body are

$$\left(1-\frac{\sinh 2\xi}{\cosh 2\xi-\cos 2\eta}\right),\qquad \log 2(\cosh 2\xi-\cos 2\eta)-2\xi,$$

and the combination of these with the first-order solution (7) that satisfies the tangency condition is

$$\chi=\beta^2 abs\left[(a^2+b^2)\left(1+\frac{3}{4}\log\frac{\beta c}{4}+\frac{3}{4}\xi-\frac{1}{4}\frac{\sinh 2\xi}{\cosh 2\xi-\cos 2\eta}\right)\right.$$
$$\left.-ab\left(1+\log\frac{\beta c}{4}+\frac{1}{2}\log 2(\cosh 2\xi-\cos 2\eta)\right)\right]. \tag{15}$$

From (8) and (9) the surface pressure coefficient is found to be

$$C_p=ab\left[2\log\frac{4}{\beta(a+b)}-2+\frac{ab}{\nu^2}\right]$$
$$+\beta^2 a^2 b^2\left[\frac{7}{2}-2\frac{a^2+b^2}{ab}+\frac{3}{2}\frac{a^2+b^2}{ab}\log\frac{4}{\beta(a+b)}\right.$$
$$\left.-2\log\frac{2}{\beta\nu}+\frac{1}{2}\left(\frac{a^2-b^2}{\nu^2}\sin 2\eta\right)^2\right], \tag{16}$$

where $\nu^2 = a^2 \sin^2\eta + b^2 \cos^2\eta$. The corresponding drag coefficient is

$$C_D = ab\left[2\log\frac{4}{\beta(a+b)} - 1\right] + \beta^2 a^2 b^2 \left[\frac{3}{2} - \frac{a^2+b^2}{ab} + \left(\frac{3}{2}\frac{a^2+b^2}{ab} - 2\right)\log\frac{4}{\beta(a+b)}\right]. \quad (17)$$

The first terms in these expressions are the previous slender-body results, and the second terms the not so-slender-corrections.

2.3. *Second-order solution neglecting triple products*

Consider next the non-linear terms in the equation of motion (3) that are products of perturbation quantities, the triple products in the last group being deferred to the next section. Thus the iteration equation is

$$\phi_{xx} + \phi_{yy} = 2M^2(\phi_x \phi_{xz} + \phi_y \phi_{yz}) + (\gamma+1)M^4 \phi_z \phi_{zz}. \quad (18)$$

In the full second-order theory (Van Dyke 1952), and hence also its slender-body counterpart, a particular integral vanishing at the Mach cone that accounts for all terms on the right-hand side except that involving $(\gamma+1)$ is given in terms of the first approximation by

$$\psi_1 = M^2\phi\phi_z = M^2 a^2 b^2 s\left(\xi + 1 + \log\frac{\beta c}{4}\right)\left(\xi + 1 + \log\frac{\beta c}{4} - \frac{\sinh 2\xi}{\cosh 2\xi - \cos 2\eta}\right). \quad (19)$$

For the remaining term, a particular integral is given, according to the general theory (Van Dyke 1956), in terms of the cross-sectional area $S(z)$, by the plane wave

$$\psi_2 = -\frac{\gamma+1}{2}\frac{M^4}{\beta^2}\left(\frac{S'}{2\pi z}\right)^2 z = -\frac{\gamma+1}{2}\frac{M^4}{\beta^2} a^2 b^2 s. \quad (20)$$

The linear term in the tangency condition (5) has been satisfied in the previous solution. Hence a complementary function χ is required such that (since $\psi_{2\xi} = 0$)

$$\psi_{1\xi} + \chi_\xi = abs\phi_z = a^2 b^2 s\left[1 - \log\frac{4}{\beta(a+b)} - \frac{ab}{a^2\sin^2\eta + b^2\cos^2\eta}\right] \quad \text{at } \xi = \xi_0. \quad (21)$$

The desired result is a combination of the three constituents of the complementary function in the previous not-so-slender problem:

$$\chi = a^2 b^2 s\left[\left\{(2M^2 - 1)\log\frac{4}{\beta(a+b)} - M^2\right\}\left(\xi + 1 + \log\frac{\beta c}{4}\right) + M^2\left\{\log\frac{4}{\beta(a+b)} - 1\right\}\left(1 - \frac{\sinh 2\xi}{\cosh 2\xi - \cos 2\eta}\right) + \beta^2\left\{\frac{1}{2}\log 2(\cosh 2\xi - \cos 2\eta) - \xi\right\}\right]. \quad (22)$$

From (8) and (9) the surface pressure coefficient is found to be

$$C_p = ab(2\lambda+\mu)+\beta^2 ab[3ab\lambda^2+\tfrac{3}{2}(a^2+b^2)\lambda-\tfrac{1}{2}(a-b)^2+\tfrac{1}{2}ab] + a^2b^2\left[(\gamma+1)\frac{M^4}{\beta^2}-2M^2\lambda+(M^2-2)\lambda\mu+(\tfrac{1}{4}M^2-1)\mu^2\right], \quad (23)$$

where $\lambda = \log\dfrac{4}{\beta(a+b)}-1, \quad \mu = \dfrac{ab}{a^2\sin^2\eta+b^2\cos^2\eta}.$

The two rather complicated last terms in the not-so-slender result (16) have here disappeared, so that this more complete result is actually easier to compute. The drag coefficient is

$$C_D = ab(2\lambda+1)+\beta^2 ab\{3ab\lambda^2+\tfrac{3}{2}(a^2+b^2)\lambda-\tfrac{1}{2}(a-b)^2+\tfrac{1}{2}ab\} + a^2b^2\left[(\gamma+1)\frac{M^4}{\beta^2}-(2+M^2)\lambda+(\tfrac{1}{4}M^2-1)\frac{a^2+b^2}{2ab}\right]. \quad (24)$$

2.4. *The effect of triple products*

The triple products appearing as the last terms in (3) are known to give contributions of second order for bodies of revolution, although they have been neglected in the few existing second-order solutions for thin wings. Their effect is found by considering the iteration equation

$$\phi_{xx}+\phi_{yy} = M^2(\phi_x^2\phi_{xx}+2\phi_x\phi_y\phi_{xy}+\phi_y^2\phi_{yy}). \quad (25)$$

This is precisely the iteration equation for the second approximation of the Janzen–Rayleigh procedure for plane subsonic flow. Moreover, the boundary conditions are essentially the same as in that problem, since the normal velocity at the surface must vanish in both cases, and it suffices here, as in the Janzen–Rayleigh problem, to require the velocity disturbances to vanish at infinity except possibly for a multiple of the first approximation. Use can therefore be made of existing treatments of the Janzen–Rayleigh problem, of which one of the most elegant is Kaplan's (1942) method of residues. That method gives the complementary function as well as the particular integral directly in terms of quadratures, which is advantageous here because, although the previous complementary functions were readily found by trial, that to be obtained next could have been guessed only with extraordinary insight.

Kaplan's method utilizes the complex variable Z in the plane in which the body appears transformed into a circle. In the present case, the elliptic cone is the unit circle in the Z plane if

$$Z = \sigma\exp(\xi+i\eta), \quad \sigma^2 = \frac{a-b}{a+b}. \quad (26)$$

The first-order slender-body solution (7) expressed in terms of Z is (following Kaplan's notation) the real part of

$$f_0(Z) = abs\left[1+\log\frac{\beta(a+b)Z}{4}\right], \quad (27)$$

and the complex velocity W_0 is given by

$$W_0(Z)=f_0'(Z)=\frac{abs}{Z}. \tag{28}$$

Then Kaplan's equation (32), without the last term (which corresponds to a uniform flow at infinity), with $U=1$, $R_1=1$, and a factor M^2 supplied, gives for the second-order complex velocity

$$\begin{aligned}W_1(Z,\bar{Z})=\tfrac{1}{2}M^2\frac{(ab)^3}{(a+b)^2}s\Bigg[&2\log\frac{1+\sigma^2}{1-\sigma^2}\left\{\frac{Z}{(1-\sigma^2Z^2)^2}-\frac{Z}{(Z^2-\sigma^2)^2}\right\}\\&+\frac{4}{1-\sigma^4}\left\{\frac{1}{Z(1-\sigma^2Z^2)}-\frac{Z}{Z^2-\sigma^2}\right\}\\&-\frac{2Z}{(Z^2-\sigma^2)(1-\sigma^2Z^2)}-\frac{1}{\sigma}\frac{1+\sigma^2Z^2}{(1-\sigma^2Z^2)^2}\log\frac{Z+\sigma}{Z-\sigma}\\&+\frac{2\bar{Z}}{(Z^2-\sigma^2)(\bar{Z}^2-\sigma^2)}+\frac{1}{\sigma}\frac{Z^2+\sigma^2}{(Z^2-\sigma^2)^2}\log\frac{\bar{Z}+\sigma}{\bar{Z}-\sigma}\Bigg].\end{aligned} \tag{29}$$

Now if the second-order increment in ϕ is the real part of $f_1(Z,\bar{Z})$, then $\partial f_1/\partial Z=W_1$ and $\partial\bar{f}_1/\partial\bar{Z}=\bar{W}_1$. Hence integration gives

$$\begin{aligned}\phi=-\tfrac{1}{2}M^2\frac{(ab)^3}{(a+b)^2}s&\\ \times\mathscr{R}\Bigg[&\frac{4}{1-\sigma^4}\left\{1+\log\frac{\beta(a+b)(Z^2-\sigma^2)}{4Z}\right\}+\frac{1}{\sigma}\frac{Z}{1-\sigma^2Z^2}\log\frac{Z+\sigma}{Z-\sigma}\\&-\frac{1-\sigma^4}{\sigma^2}\log\frac{1+\sigma^2}{1-\sigma^2}\frac{Z^2}{(Z^2-\sigma^2)(1-\sigma^2Z^2)}+\frac{Z}{\bar{Z}^2-\sigma^2}\log\frac{\bar{Z}+\sigma}{\bar{Z}-\sigma}\Bigg].\end{aligned} \tag{30}$$

The constant of integration has been chosen so that this behaves far from the body like the first-order solution (27). The last term inside the bracket is the particular integral, and the remainder is the complementary function, which is very complicated when written in terms of real variables. Some computation yields the following increment in surface pressure due to triple products which is to be added to (23):

$$\begin{aligned}\Delta C_p=M^2a^2b^2\Bigg[&1-\log\frac{2}{\beta\nu}\\&+\frac{a^2b^2}{\nu^2(a^2-b^2)}\left(\frac{a^2}{\nu^2}\frac{\sqrt{\nu^2-b^2}}{b}\cos^{-1}\frac{b}{\nu}+\frac{b^2}{\nu^2}\frac{\sqrt{a^2-\nu^2}}{a}\cosh^{-1}\frac{a}{\nu}-\log\frac{a}{b}\right)\\&+\left(\frac{a^2-b^2}{2\nu^2}\sin 2\eta\right)^2+\frac{a^2b^2(a^2-b^2)}{2\nu^6}\sin^2 2\eta\\&\times\left(\log\frac{a}{b}-\frac{\sqrt{\nu^2-b^2}}{b}\cos^{-1}\frac{b}{\nu}-\frac{\sqrt{a^2-\nu^2}}{a}\cosh^{-1}\frac{a}{\nu}\right)\Bigg],\end{aligned} \tag{31}$$

where again $\nu^2=a^2\sin^2\eta+b^2\cos^2\eta$. The integrals involved in calculating the drag aro treated by contour integration of $(Z^2-\sigma^2)^{-1}\log(1+\sigma Z)/(1-\sigma Z)$ around the unit circle, together with integration by parts. Thus the increment in drag due to triple products, which is to be added to (24), is found to be

$$\Delta C_D=M^2a^2b^2\left[\frac{3}{8}\frac{a^2+b^2}{ab}-\log\frac{4}{\beta(a+b)}+\frac{3}{2}\frac{ab}{a^2-b^2}\log\frac{a}{b}\right]. \tag{32}$$

2.5. *Reliability and accuracy of the solution*

Setting b equal to a yields the second-order slender-body solution for a circular cone, which agrees with Broderick's result (1949).

As a check, an independent second-order solution (based upon the Prandtl–Glauert equation without the slender body-approximation) has been carried out for a slightly eccentric elliptic cone by perturbing the solution (Van Dyke 1952) for a circular cone. Three terms were retained in an expansion in powers of the eccentricity, and the resulting surface pressure and drag, when expanded into slender-body form, agree completely with the expansion of the above solution in powers of the eccentricity. This check, which is so elaborate as to be almost conclusive, was applied also to the not-so-slender solution. As an additional check, the pressure has been studied in the vicinity of the leading edge as the elliptic cone collapses to a flat wing. In this limit the leading edge approaches a thin highly yawed parabolic cylinder, and the above solution is found to reproduce the second-order Janzen–Rayleigh solution (Imai 1952) for a parabola in the subsonic flow corresponding to the component of free-stream velocity normal to the edge.

Despite occasional statements to the contrary, slender-body theory is generally somewhat less accurate than linearized theory (of which it is a further simplification), particularly at high supersonic speeds, and the inferiority is compounded in higher approximations (Van Dyke 1952). However, at 'ordinary' supersonic Mach numbers, of which the archetype is $M = \sqrt{2}$, the first two approximations are known to approach rapidly the exact solution for reasonably slender cones (Broderick 1949), and comparable accuracy is to be expected for elliptic cones subtending similar solid angles.

Recently, Rogers & Berry (1956) have measured surface pressures at a Mach number of 1·41 over two flat elliptic cones having $a = \tan 30° = 0{\cdot}577$ and $b = 0{\cdot}05$ and 0·10. Figures 2 and 3 show their measurements at zero angle of attack, which agree well with the present theory. Pressures are plotted against $\eta = \cos^{-1}(x/as)$ in order to expand the narrow region of high pressure near the leading edge. Also shown for the thinner wing in figure 2 is the result of neglecting triple products, which is the second-order theory used by Rogers & Berry. Inclusion of triple products is seen to yield significant improvement in a small region behind the leading edge. The maximum increment in pressure coefficient due to triple products is 0·036, whereas that due to double products is $-0{\cdot}058$. For the thicker wing the corresponding maximum values are actually smaller, being 0·027 and $-0{\cdot}032$, but the maximum net effect of both kinds of non-linear terms is somewhat greater than that for the thinner wing.

These two wings are equivalent in cross-sectional area to circular cones having semi-vertex angles of 9·6° and 13·5°, for which second-order slender-body theory predicts pressure coefficients only a few per cent too high (Broderick 1949). However, it might be feared that, with $\beta a = 0{\cdot}577$, the planform is so wide that the error in the slender-body expansion would be much greater. At least in the limit of vanishing thickness it can be shown that this fear is unwarranted. The surface pressure almost everywhere is given exactly in the limit by Squire's (1947) linearized solution, and second-order slender-body theory is 5·7% low. At the leading edge the pressure rises to a maximum whose exact value in the limit of zero thickness corresponds to loss of the normal component of free-stream velocity, and second-order slender-body theory is there only 1·8% low. (The corresponding defects in first-order slender-body theory are 18·8% and 15·2%.) Thus the good agreement between theory and experiment shown in figures 2 and 3 might reasonably have been expected.

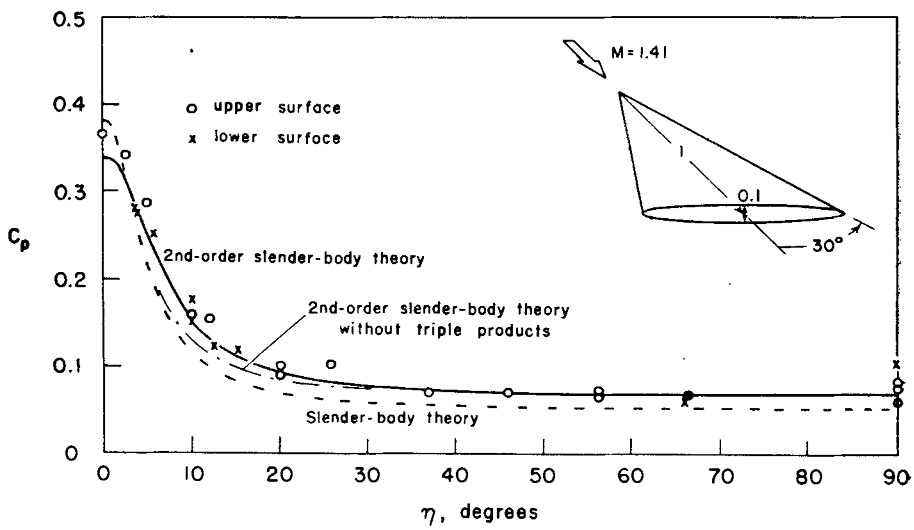

Figure 2. Theoretical and experimental pressures over 10% thick elliptic cone.

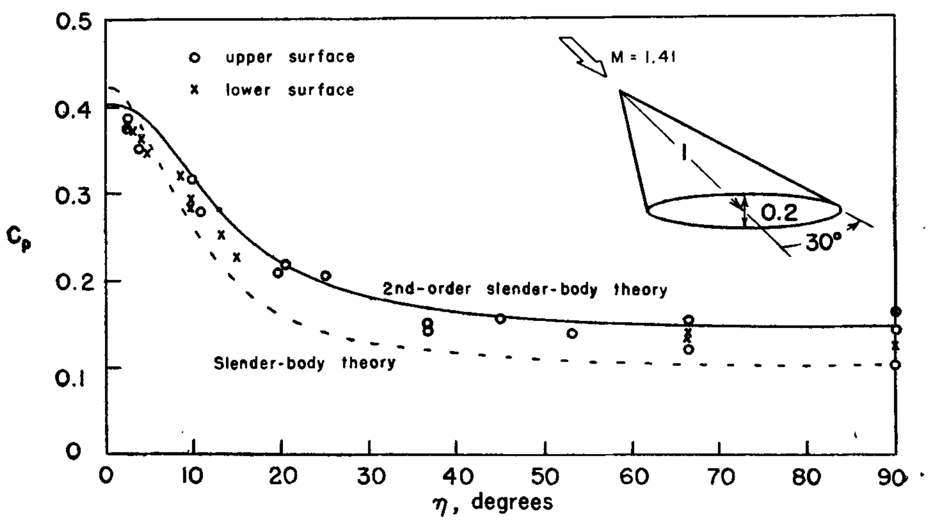

Figure 3. Theoretical and experimental pressures over 20% thick elliptic cone.

3. A critical discussion of other approximations

3.1. *Slender-body and linearized theory*

The above results are now to be used as a model of the full solution, so that other approximations will be considered as additional to those already made. The second-order slender-body solution will therefore be referred to as the 'exact' solution. On this basis, the not-so-slender approximation is the ' linearized ' approximation.

In this model, as in the full theory, expansion of the linearized solution in powers (and logarithms) of the body thickness yields the slender-body solution as the leading terms. Slender-body, linearized, and exact theory have frequently been compared for circular cones (e.g., Broderick 1949). Consider, therefore, the other extreme of flat cones, as exemplified by the wing of figure 3. For convenience, all comparisons are made at $M=\sqrt{2}$. The surface pressures predicted by slender-body, 'linearized', and 'exact' theory are compared in the left half of figure 4. Within this model, linearized theory is seen to be definitely superior to slender-body theory. (It becomes even more accurate if one uses the linearized velocity components in the full isentropic pressure-velocity relation.)

3.2. *First-order thin-wing theory*

Thin-wing theory is a simplification of linearized theory in which the condition of tangential flow is imposed not at the surface but at (say) the plane $y=0$. Again, in the second and higher approximations the tangency condition is transferred to that plane by Taylor series expansion. It is well known that, as a consequence of this planar approximation, the solution is not uniformly valid, in particular along the edges of the wing, where singularities arise that are intensified in higher approximations.

The thin-wing solution for an unyawed elliptic cone lying inside the Mach cone in supersonic flow was given by Squire (1947), who found the surface pressure to be constant. The value involves complete elliptic integrals, but two terms of their series expansions yield the model required here:

$$C_p = 2ab\left(\log\frac{4}{\beta a} - 1\right) + \beta^2 a^3 b\left(\frac{3}{2}\log\frac{4}{\beta a} - 2\right). \tag{33}$$

The same result is, of course, obtained by expanding formally the not-so-slender solution (16) for small thickness b and retaining only linear terms. This two-term approximation is within three per cent of Squire's result for semi-apex angles as great as half the Mach angle, whereas the first term alone (the thin-wing limit of slender-body theory) gives that accuracy only out to one-fifth of the Mach angle. Comparison with the 'exact' pressure distribution is shown on the right half of figure 4.

Integration of this pressure over the surface would give a drag coefficient equal to the right-hand side of (33). However, R. T. Jones has pointed out (1950) that this procedure misses a term associated with the singularity at the edge. His full expression for the additional leading-edge drag of a flat elliptic cone, and its representation in the present model, are

$$C_{D_{L.E.}} = \frac{ab}{\sqrt{(1-\beta^2a^2)}} \sim ab + \tfrac{1}{2}\beta^2a^3b. \tag{34}$$

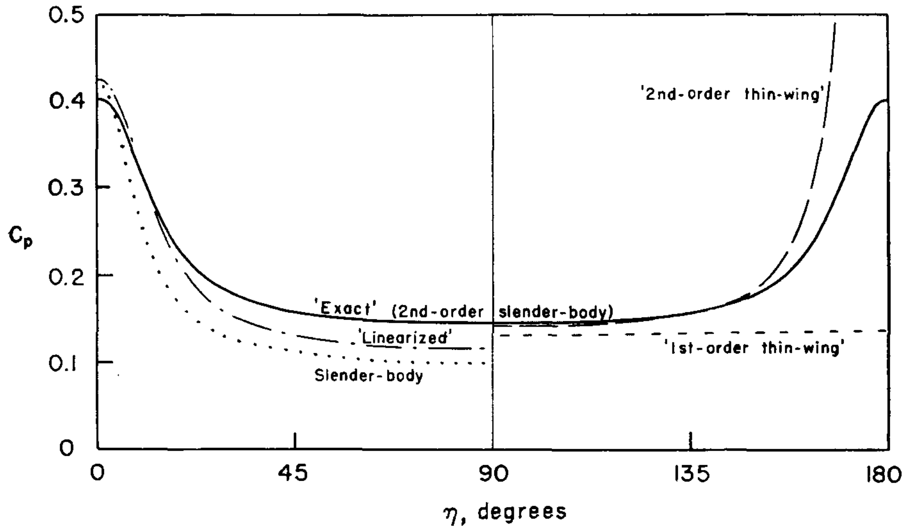

Figure 4. Effect of further approximations upon second-order slender-body solution for elliptic cone of figure 3.

Adding this to (33) gives

$$C_D = ab\left(2\log\frac{4}{\beta a} - 1\right) + \beta^2a^3b\left(\frac{3}{2}\log\frac{4}{\beta a} - \frac{3}{2}\right). \tag{35}$$

This agrees with the result of expanding the 'exact' drag (24 + 32) for small b. However, expanding the 'linearized' drag (17) gives -1 instead of $-\frac{3}{2}$ in the last term. This discrepancy indicates that linearized theory (even with tangency imposed at the actual surface) fails to predict the leading-edge drag correctly. Non-linear terms are required, which are implicit in Jones's expression (34), although he derived it by an ingenious use of linearized theory.

3.3. *Second-order thin-wing theory*

Formal expansion of the 'exact' pressure (23 + 31) for small thickness and retention of squares of b yields the 'second-order thin-wing' result

$$C_p = 2ab\left(\log\frac{4}{\beta a} - 1\right) + \beta^2a^3b\left(\frac{3}{2}\log\frac{4}{\beta a} - 2\right) + b^2\left[\frac{1}{\sin^2\eta} - 2 + 3\beta^2a^2\left(\log\frac{4}{\beta a} - 1\right)^2 + M^2a^2\left(\frac{1}{\sin^2\eta} + 2 - 2\log\frac{4}{\beta a} - \log\frac{2}{\beta a\sin\eta} + (\gamma+1)\frac{M^2}{\beta^2}\right)\right]. \tag{36}$$

Fenain & Germain (unpublished) have recently calculated the full second-order thin wing-solution for the elliptic cone. When reduced to the present approximation, their expression for pressure agrees with (36) except for the terms $M^2a^2b^2[(\sin\eta)^{-2}-\log(2/\beta a\sin\eta)]$ arising from the triple products, which they do not consider.

The contribution of the triple products is of the same order as the other second-order terms, and, as pointed out above, the numerical values are of the same magnitude. Hence it appears that triple products must be included in any complete second-order thin-wing solution for a wing having round subsonic edges.

Several of the second-order terms in (36) are infinite at the leading edge ($\eta=0$), so that the pressure distribution has the form shown on the right in figure 4. The singularity is non-integrable, and must therefore be eliminated before the pressure can be integrated to find the drag. Corrections of this sort are discussed below.

Moore (1950) has given the full second-order thin-wing solution for a non-lifting cone of diamond section lying inside the Mach cone. For narrow planforms and very small thickness he finds large second-order effects over the entire wing surface. Lighthill has suggested (1954) that this may indicate complete breakdown of the planar approximation for narrow wings having stagnation edges. However, the round edges of the present example are a more severe test of the planar approximation, and the moderate magnitude of second-order effects shown in figure 4 (except for the inevitable leading-edge singularity) suggests rather that Moore's computations are in error. This conclusion is confirmed by Fenain & Germain's recent recalculation (1955) of Moore's problem, which has uncovered an error in his analysis.

The 'first-order' pressure coefficient (33) has the form $b\beta^{-1}F_0(\beta a, \beta b, \eta)$ consistent with the generalized Prandtl–Glauert rule, and the 'second-order' increment in (36) has the form $b^2[F_1+M^2\beta^{-2}F_2+(\gamma+1)M^4\beta^{-4}F_3]$ found by Fenain & Germain (1955) in their treatment of the flat diamond cone.

3.4. *Edge corrections*

Recently the author has proposed rules for rendering thin-wing theory uniformly valid at subsonic edges (Van Dyke 1954). The rules for round-nosed airfoils are based, to second order, upon consideration of subsonic flow past a parabola having the same radius as the edge, and consist essentially in multiplying the thin-wing solution by the ratio of the exact velocity on the parabola to its thin-wing expansion. It was suggested that for swept edges the rules are to be applied to the normal component of velocity.

In the present model, the 'exact' solution for a parabola in subsonic flow consists of the first two terms of the Janzen–Rayleigh approximation (Imai 1952). Applying the resulting correction to the 'first-order thin-wing' solution (33) yields the uniformly valid first approximation

$$\bar{C}_p = \text{`}C_p\text{'} + \frac{a^2(1-a^2)}{1+\tau} + (M^2-2)\frac{a^3b}{1+\tau}\left(\log\frac{4}{\beta a}-1\right) + \frac{a^4M^2}{4(1+\tau)^2} + \frac{a^4M^2\tau}{(1+\tau)^2}\left[1-\frac{1}{1+\tau}\log\frac{4}{1+\tau}+\frac{1-\tau}{1+\tau}\,\frac{\tan^{-1}\sqrt{\tau}}{\sqrt{\tau}}\right], \quad (37)$$

where

$$\tau = \frac{2}{b^2}\,\frac{1\mp\cos\eta}{a^{-2}\pm\cos\eta}.$$

Here 'C_p' is the formal thin-wing result (33); and the upper and lower signs apply to the edges where $\eta = 0$ and π respectively. This approximation can scarcely be distinguished from the 'exact' solution to the scale of figure 4; the comparison is therefore shown in the following table:

η, degrees	0	5	10	20	90
'Exact' C_p, (23+31)	0·404	0·380	0·317	0·220	0·145
Uniform 1st approx. (37)	0·403	0·370	0·305	0·212	0·132

The refinement of including effects of both edges, not made here, would improve the agreement, giving, for example, 0·138 instead of 0·132 at $\eta = 90°$.

An unexpected complication arises when the corresponding correction is undertaken for the 'second-order thin-wing' solution (36). Although the algebraic singularity is removed, the logarithmic singularity persists in the component of velocity tangential to the leading edge, and hence also in the pressure. The result is therefore not uniformly valid. This means that application of the two-dimensional correction to the normal component of velocity at a swept edge, which was thought (Van Dyke 1954) to be evidently valid, is incorrect for round edges in the second approximation except for incompressible flow. A re-examination of this problem is necessary.

3.5. *Linear perturbation of flow past circular cone*

Ferri and his colleagues have made considerable use of the 'linearized characteristics method', which consists in linearizing in the departure from some known basic flow other than a uniform stream. Thus non-circular cones are treated by linear perturbation of the known solution for circular cones. Although this approximation implies that the cross-section deviates only slightly from a circle, Ferri (1951) has reported good agreement with experiment for an elliptic cone of 3 : 1 axis ratio.

The present model has been treated by following Ferri's procedure of expanding the velocity components in Fourier series in the polar angle θ, linearizing consistently with respect to deviation of the cross-section from circular, but then calculating pressure from the full relation, which is (9) here. The left half of figure 5 compares the resulting pressures with the 'exact' solution for an elliptic cone of 3 : 1 axis ratio and area equivalent to a 10° circular cone, at $M = \sqrt{2}$. The rather large discrepancy can be understood by considering the corresponding expansions in the still simpler case of slender-body theory. Since $\tan\eta = (a/b)\tan\theta$, the slender-body pressure (10) can be shown to have the Fourier expansion in θ

$$C_p = 2ab\left[\log\frac{4}{\beta(a+b)} - 1\right] + \frac{2a^2b^2}{a^2+b^2}\sum_{n=0}^{\infty}\left(\frac{a^2-b^2}{a^2+b^2}\right)^n \cos 2n\theta. \qquad (38)$$

Termination of this series at $\cos 10\theta$ introduces a considerable error, as shown on the right half of figure 5. An additional error of roughly the same magnitude is seen to arise from linearization in the departure from the

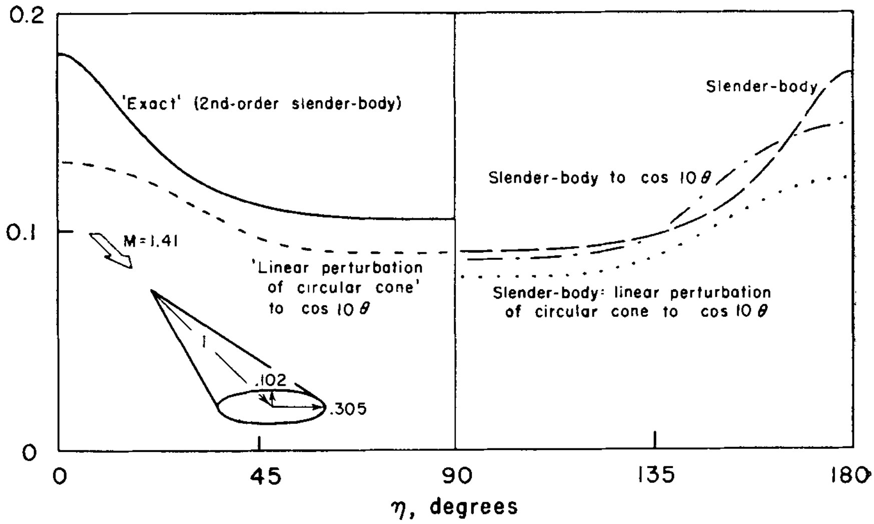

Figure 5. Accuracy of method of linear perturbation of flow past circular cone.

solution for a circular cone (which in this case affects only the surface boundary condition, since the equation of motion is already linear). The latter error might be reduced by retaining non-linear terms in the boundary condition, as has been suggested (Ferri, Ness & Kaplita 1953). However, the remaining error inherent in curtailing the Fourier series is so great that if only terms up to $\cos 10\theta$ are to be retained, it would seem that the method must be restricted to more nearly circular bodies.

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